Introduction to Fixed Income

Term Structure Modelling

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Outline

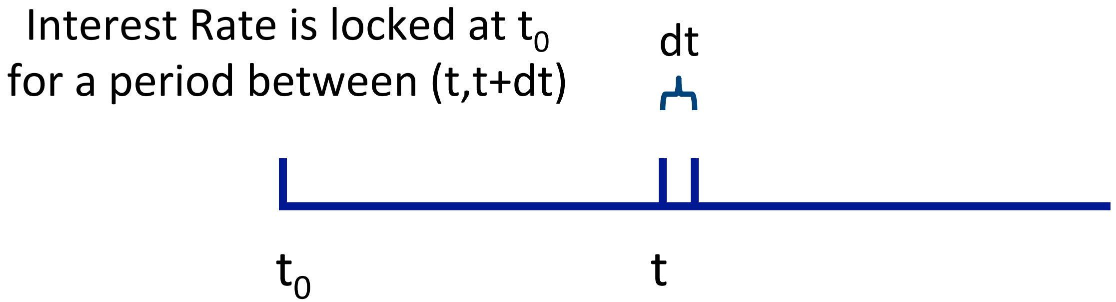
- Short Rate
  - Idea
  - Instantaneous Forward Rate
  - Parity Formula
- Term Structure Modeling
  - Short Rate Model
  - Pricing Bonds with Embedded Options

Short Rate

- Question:
  - Suppose L wants to lend $100 to B for one second.
  - What would be the interest?
  - What would be the interest rate?
- Solution:
  - I=PRT, As T is almost 0, Interest would be almost 0.
  - There is nothing which suggests that the interest rate would be 0. In fact this interest rate would be a non zero quantity.
- Short Rate: is the interest rate at which one can borrow money for an infinitesimally short period of time.
- Short Rate is the spot rate for an infinitesimally small maturity.

Instantaneous Forward Rates

- Instantaneous Forward Rate for time t is the forward rate that can be locked now for a loan between time ‘t’ and time ‘t+dt’.
- Instantaneous Forward Rate ‘locks in’ the short rate for some future time t.



- Short Rates and Instantaneous Forward Rates are used extensively in modelling Term Structure.
- There is a one to one mapping between Spot Rate Curve and Instantaneous Forward Rate Curve.

Short Rate-Instantaneous Forward Rate

- Short Rate is the spot rate for the next instant.
- Short Rate 5 years from now would be the spot rate for a loan which would start at 5 years and end at the next instant.
- Instantaneous Forward Rate for a date is 'locked in' rate for the future short rate for that date.

Price of Petrol (today)	Price of Petrol (6months from now)	Forward on Price of Petrol 6months from now
Known Today	Known 6 months from now	Known Today

- Replace Price of Petrol with Short Rate and one can understand the relationship between the two.

Spot Rate- Instantaneous Forward Rate

- Instantaneous Forward rate can be found as a limiting case of forward rate with $t_1 \rightarrow t_2$.

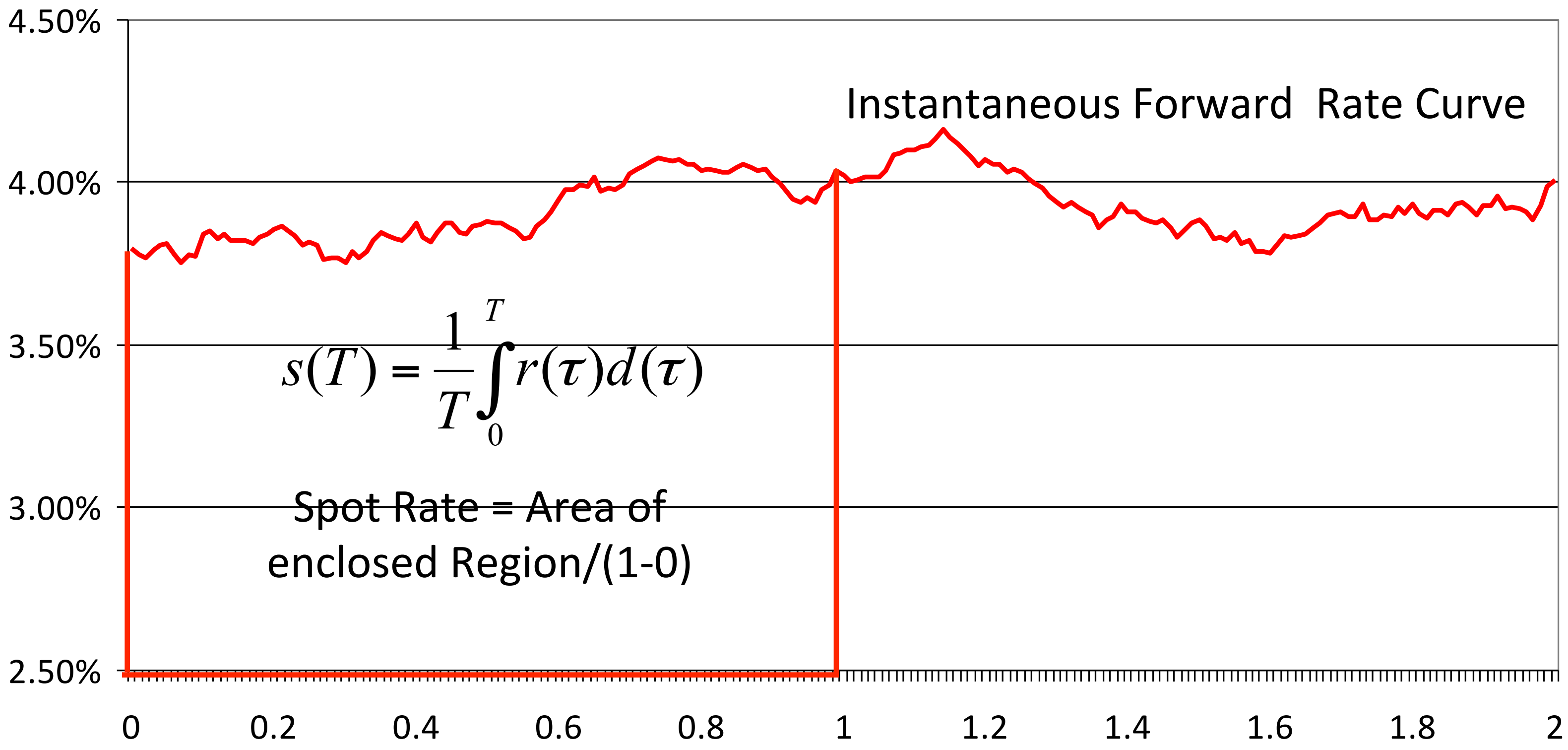
$$f_{12} = \frac{s_2 t_2 - s_1 t_1}{t_2 - t_1}$$

$$t_1 = t, t_2 = t + \in$$

$$f(t) = \lim_{\in \to 0} \frac{s_{t+\in}.(t+\in) - s_t.t}{(t+\in) - t} = \frac{d}{dt}(st) \Rightarrow$$

$$s(t) = \frac{1}{T}\int_0^T f(\tau)d(\tau)$$

Instantaneous Forward Rates



Short Rate: Advantages

- Central Bank controls the economy by changing the Short Term Interest Rates.
- Central Bank does not usually have any direct control over long term interest rates.
- Investors bet on long term interest rates by speculating future short term interest rates.

$$s(t) = \frac{1}{T}\int_0^T r(\tau)d(\tau)$$

- US 10 year Spot Rate is presently 2% which implies that expectation of average short rate for next 10 year is 2%.
- Short Rate is also used for pricing callable bonds.

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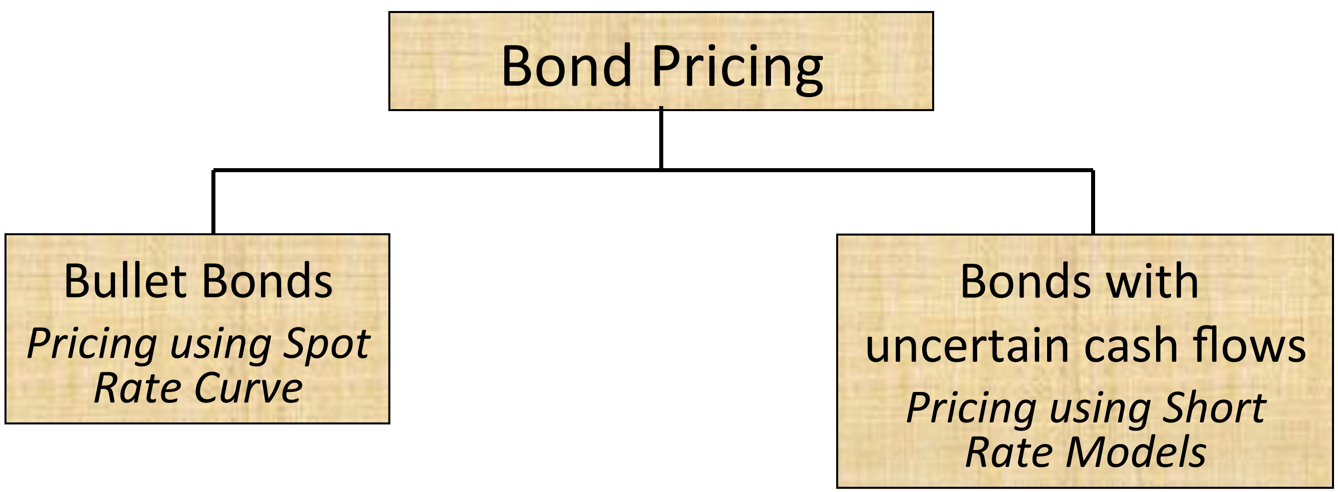
- Term Structure Modeling
  - Short Rate Models
  - Pricing Bonds with Embedded Options

Interest Rate Curve Models

- Pricing of a financial instrument is the fundamental problem of finance.
- Bonds with fixed cash flows are priced by discounting cash flows using the Spot Rate Curve.
- Bonds with uncertain cash flows are priced by discounting cash flows using various possible interest rate curves.

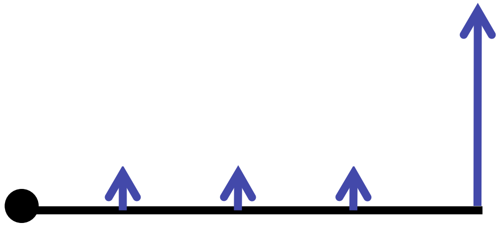


Pricing Callable Bonds

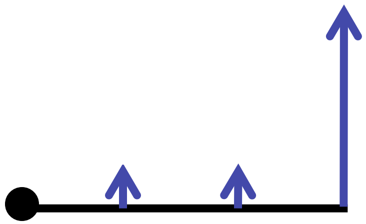
- Callable Bonds are bonds which can be ‘called’ (prepaid) by the issuer.
- Reliance has issued a callable bond (taken a loan) for 2 years for a coupon of 15%.
- Call Feature allows Reliance to call the bond (at coupon payment dates) in case the interest rates goes down in future.
- Depending on the future interest rates, bond cash flows can be decided.
- Bond Price is the present value of future cash flows but in this case future cash flows are not known with certainty!

Pricing Callable Bonds

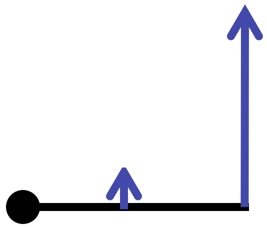
- Case A:
  - Interest Rates always remain high and the bond is never called



- Case B:
  - Interest Rates decrease at 1.5 years and the bond is terminated at 1.5 years.



- Case C:
  - Interest Rates decrease at 1 year and the bond is terminated at 1 year.



- Case D: Other similar cases

Pricing Callable Bonds

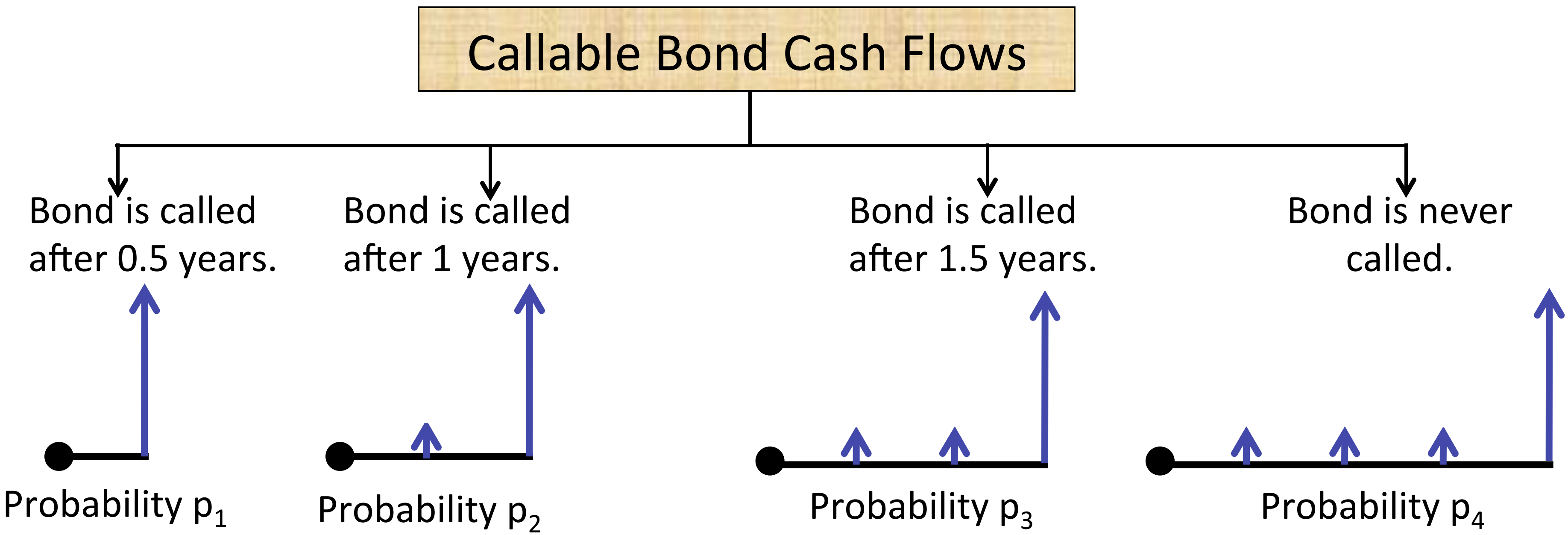
- Callable Bonds have random cash flows.
- *Price for such case is the expected value of cash flows obtained from it*

- Suppose there is a game in which a fair dice is rolled.
  - If the outcome (x) is less than equal to 4, one receives Rs 0.
  - Else he receives the difference of the outcome from 4. (x-4)
  - What price one should pay for playing this game?1

Price = Expected (Max(0,x-4))
= 1/6.(0+0+0+0)+1/6.(1+2)
= 0.5

Pricing Callable Bonds

- Expected Value of the payoffs needs to be found for which the probability distribution of the payoff is required.



- Price = Average (p_i*Price of i^{th} bond))
- Price of each bond has to be found using appropriate rate!

Pricing Callable Bonds

- How do we find the probability that the bond would terminate in (say) 1.5 years?
  - It depends on how the spot rates evolve in future.
- How do we model the spot rates?
  - Spot Rates can be modeled assuming some random process for spot rates.
- How do we take care of the fact that spot rates are not a scalar quantity which can be modeled but consists of infinite values for infinite possible maturities?
  - Spot Rate can be modeled using Short Rate which is scalar.
- Even then, can we compute the expected value?

Pricing: Monte Carlo Simulations

- Expected Value cannot always be easily calculated.
- Monte Carlo Simulations is an easy way to calculate the price without computing the expectation.
- Suppose there is a game in which coin is tossed 4 times and you earn as much as the number of tails.
- What should be the price for this game?

- Solution:
  - There are 5 possible payoffs: 0, 1 , 2, 3, 4
  - Probability of each: 0.0625, 0.25, 0.375, 0.25, 0.0625 (How?)
  - Price = Expected Value = 2
- What if I am not good at calculating these probabilities?

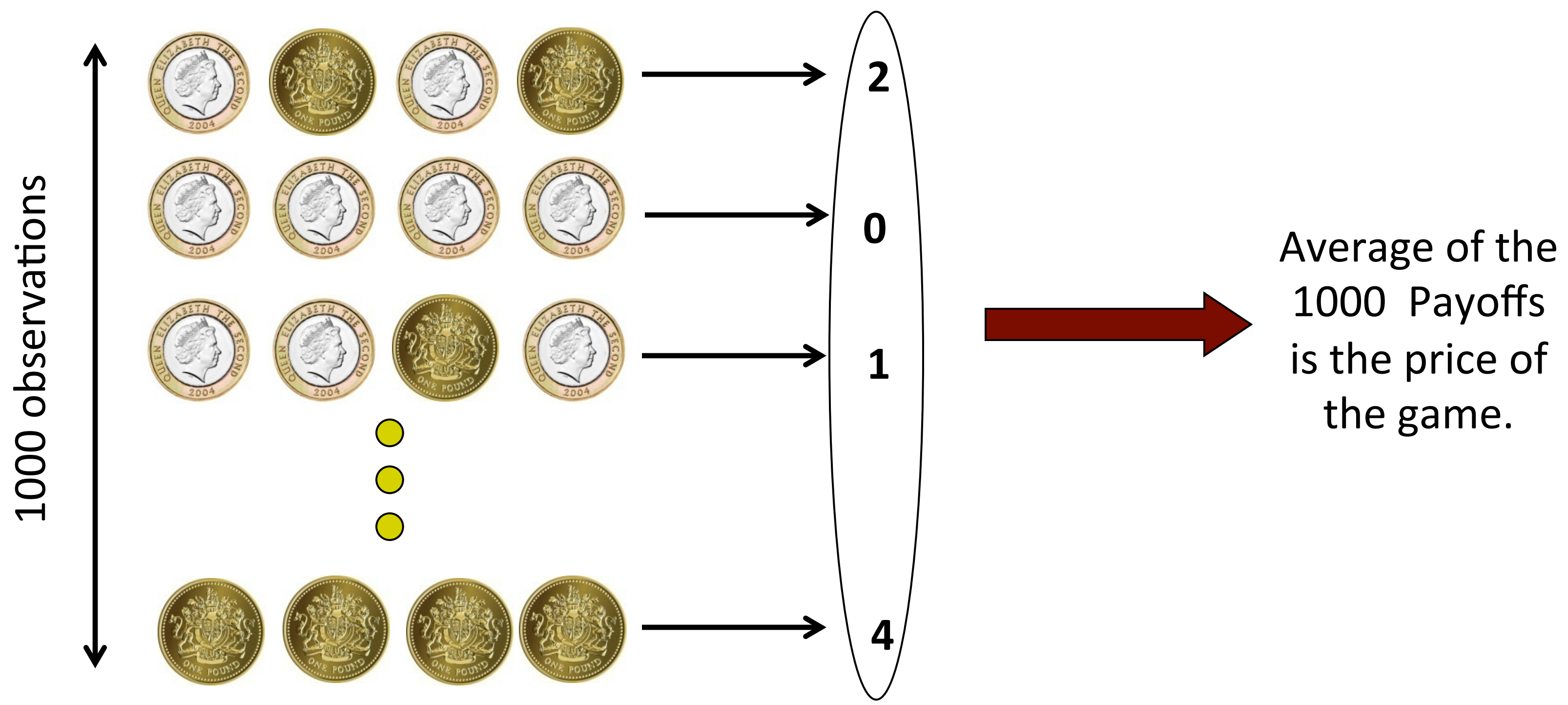
Pricing: Monte Carlo Simulations

- Expected Value is essentially the average if the experiment is undertaken infinite times.
- Why not play the game many times and find the average?
- In Monte Carlo Simulations, the 'game' is played many times and the average payoff is outputted as the price.
- In case the payoff is in future, appropriate discounting is done.

- Solution:
  - Toss the coin 4 times and observe the payoff.
  - Repeat this experiment large (say 1000) times.
  - Average of the payoffs is the price of the game.

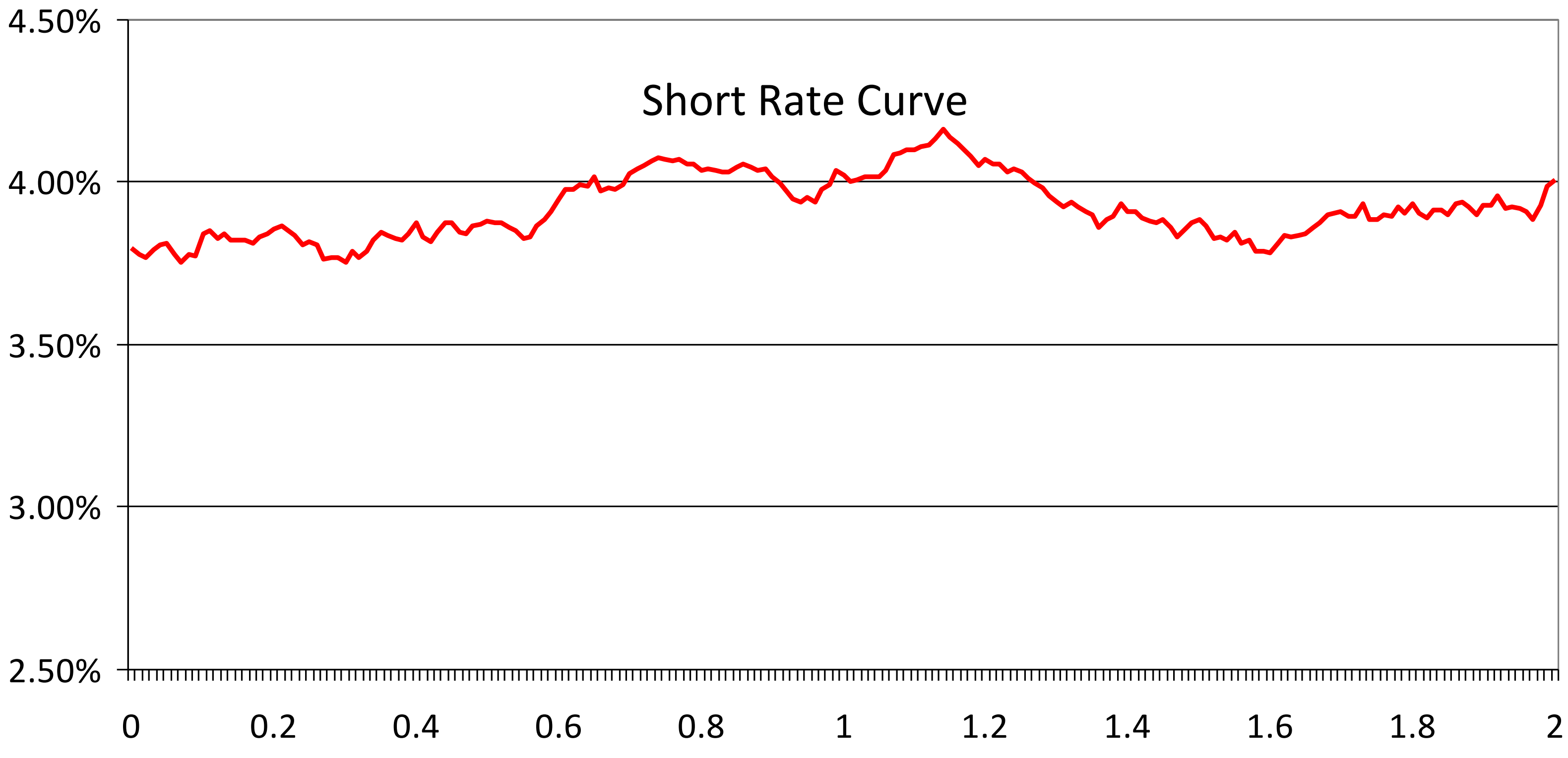
Pricing: Monte Carlo Simulations

- Suppose the game is played 1000 times.



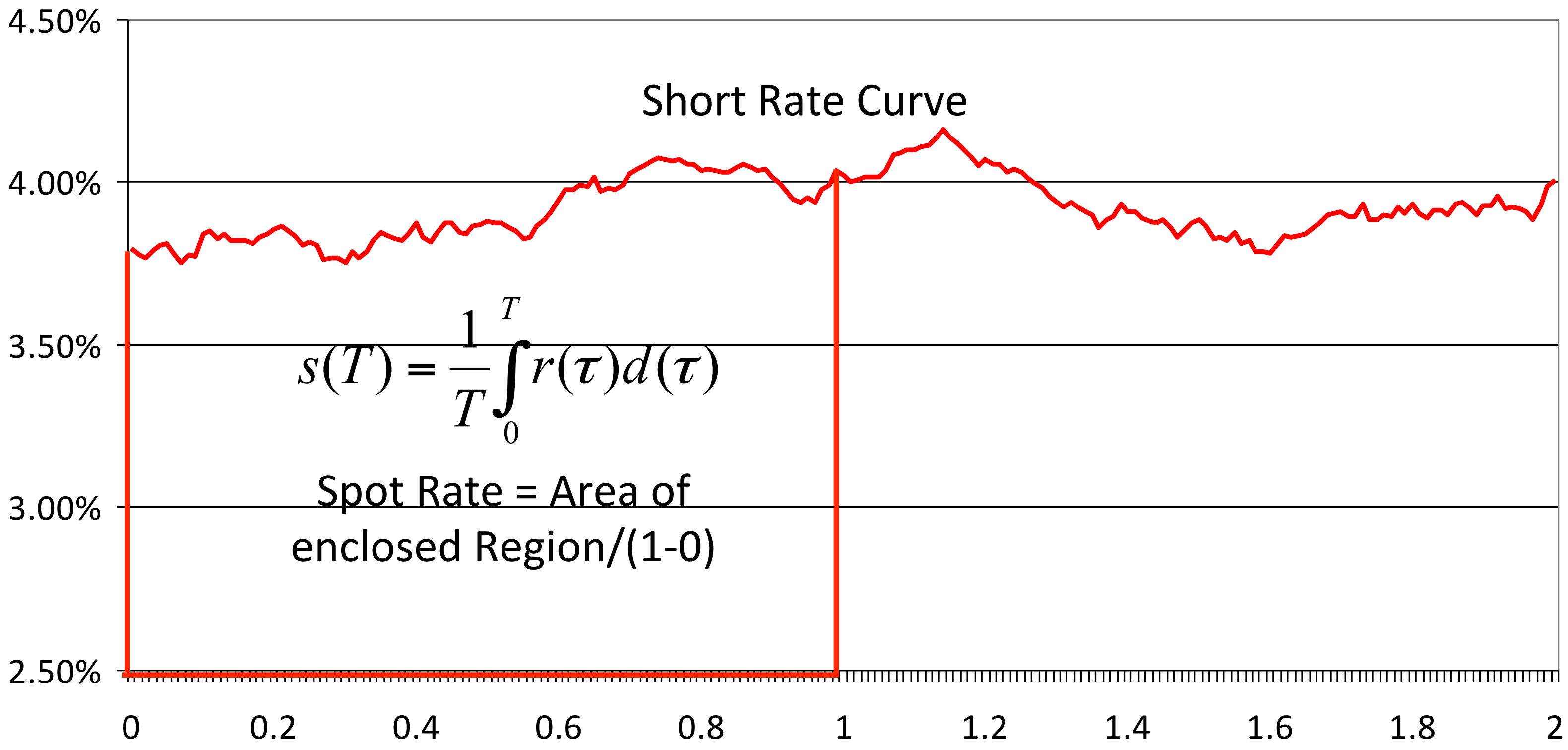
Short Rate Simulation

- Suppose the Short Rate is simulated for the next 2 years.



Short Rate Simulation

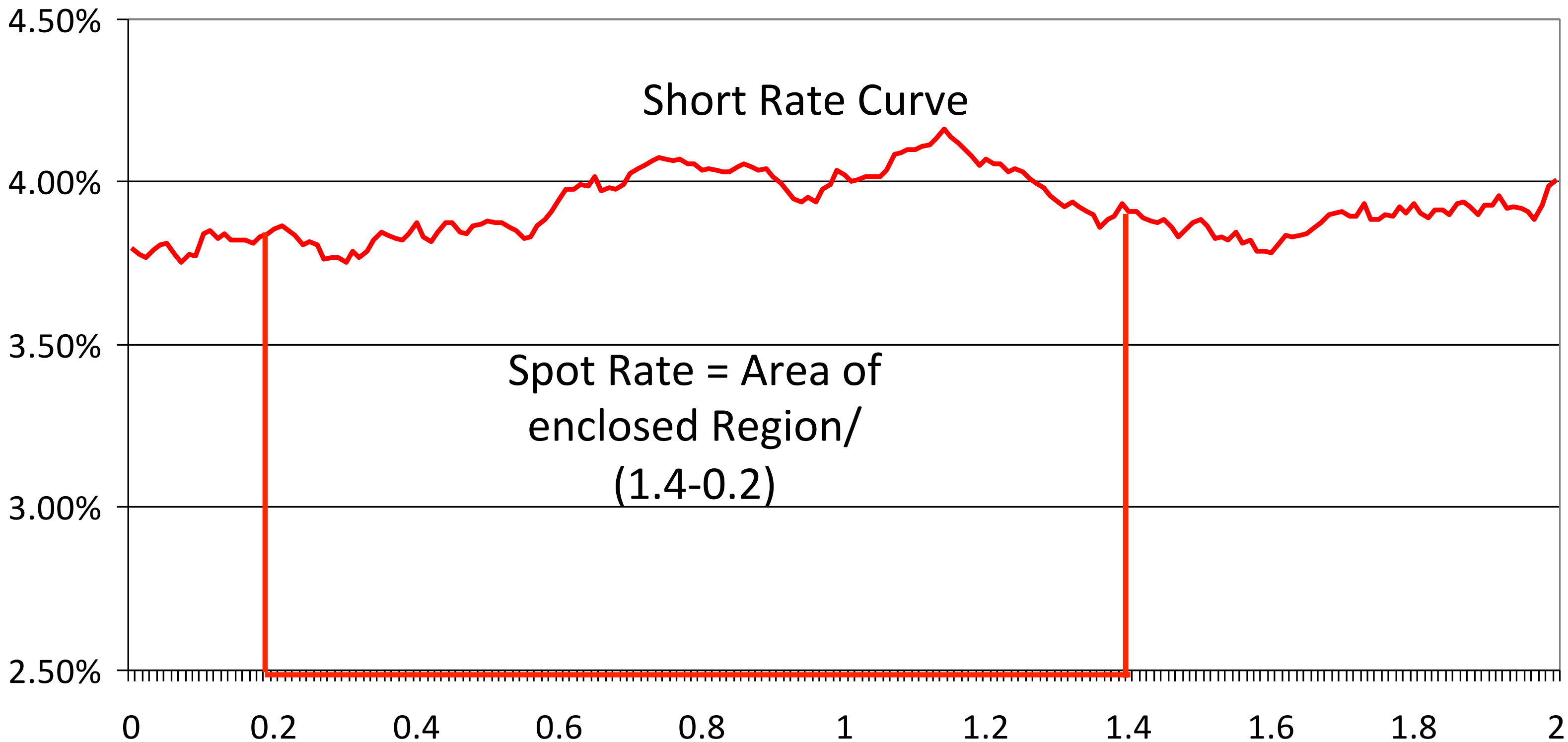
- Can we find the spot rate for 1 year?



$$s(T) = \frac{1}{T}\int_0^T r(\tau)d(\tau)$$

Short Rate Simulation

- Can we find the spot rate for 1.2 years starting on 0.2 year?



- Short Rate Curve not only tells about all the Spot Rates at present but also the Spot Rates for all maturities at all future instants.

Short Rate Simulation

- Suppose short rates are replaced by the prices of oranges.
- At present, we don't know what would be the daily prices of oranges for next one year.
- Still various possibilities of the price path of oranges can be forecasted.
- Example:
  - Price(t+1)=Price(t)+x
  - x can be -1, 0 or 1 with equal probability
- Sample Paths
  - 5, 6, 6, 7, 6, 5, …
  - 5, 4, 3, 4, 4, …
  - 5, 5, 5, 6, 7, …

Short Rate Simlulation

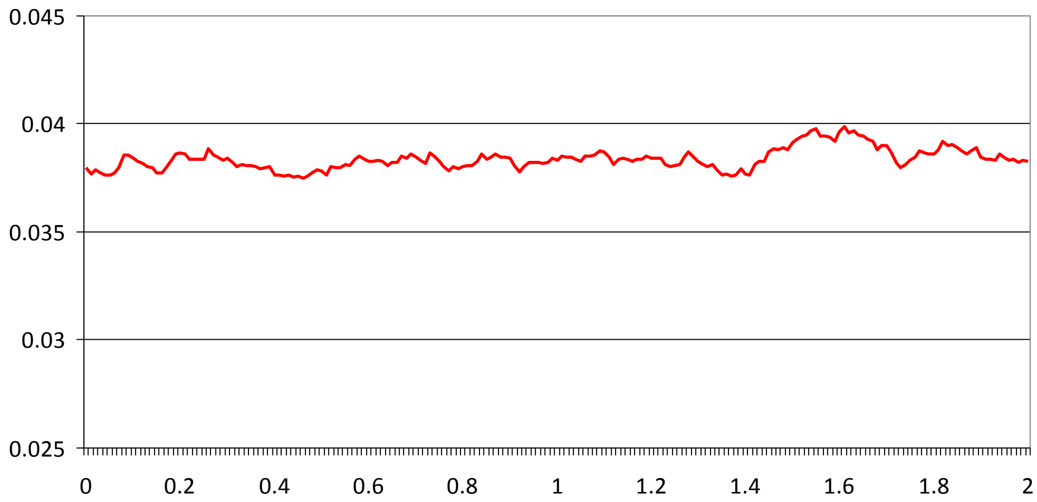
- Similarly Short Rates can be forecasted using such an equation.
- Equations with a random term defined on small differences are called Stochastic Differential Equations.

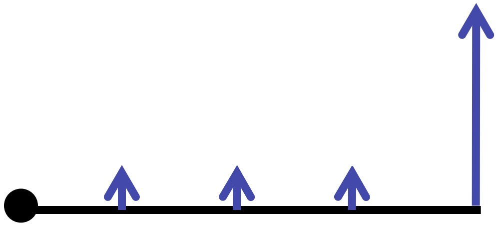
- Examples:
  - Vasicek Model $dr_t = a(b - r_t)dt + \sigma dW_t$
  - Hull White Model
  - CIR Model

- Short Rate can be simulated (Monte Carlo!) using these models.
- Models are calibrated depending on the present rates.
- Note: Short Rate Models follow risk neutral pricing!

Pricing Callable Bonds

- For each path, find the cash flow (as per the corresponding spot rate curve) and find the present value of the bond.

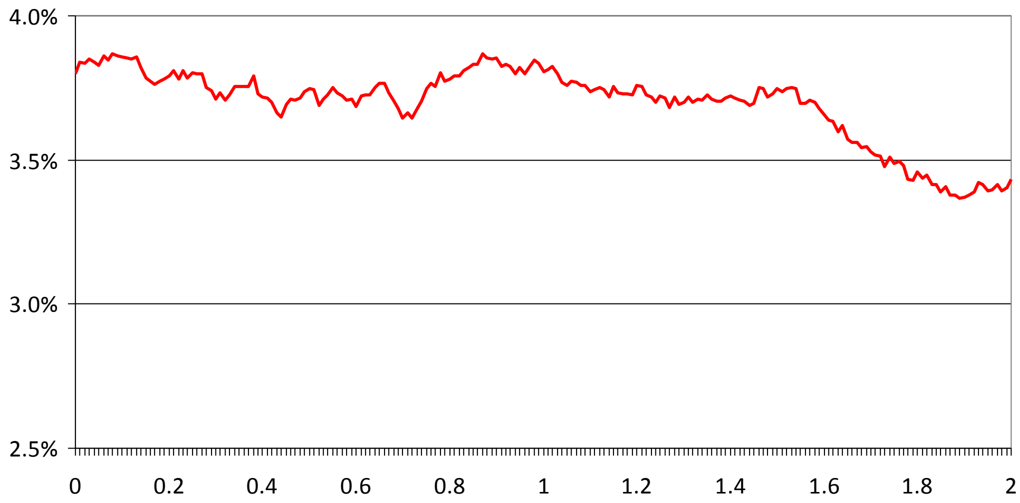


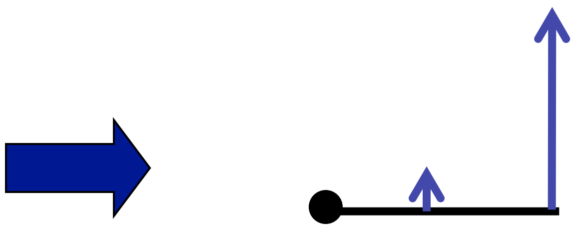


Find the price of the resultant bond.

Pricing Callable Bonds

- For each path, find the cash flow (as per the corresponding spot rate curve) and find the present value of the bond.





Find the price of the resultant bond.

- Average of price obtained in such simulations is the price of the Callable Bond!

Questions

Quiz